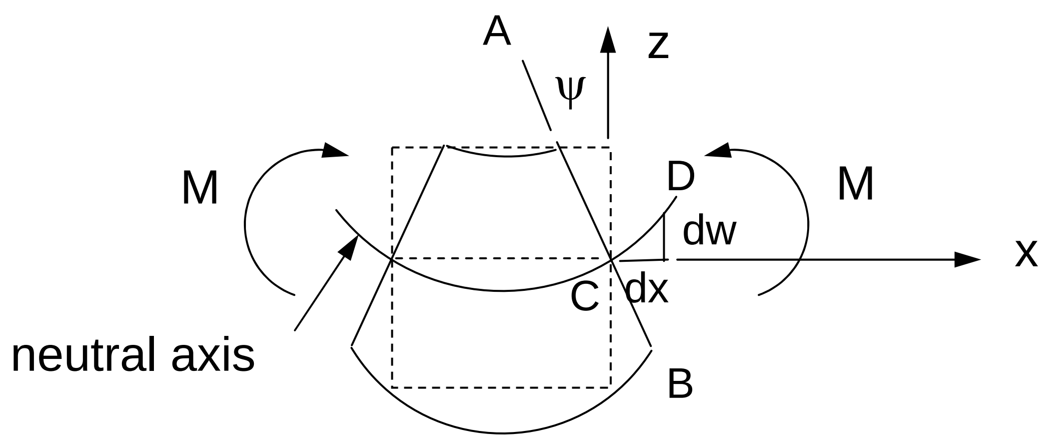
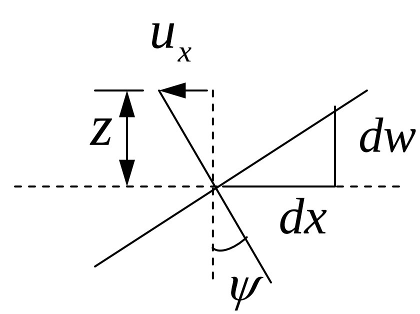
Euler-Bernoulli Bending Theory (Pure Bending Moment)



u_z = w(x) = vertical deflection of the neutral axis

$$u_x = -z\psi(x)$$



If the plane AB remains perpendicular to CD $\quad \psi = \dfrac{dw}{dx}$

$$u_x = -z\frac{dw}{dx}$$

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$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -z\frac{d^2 w}{dx^2}$$

If we assume $\quad \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = 0$

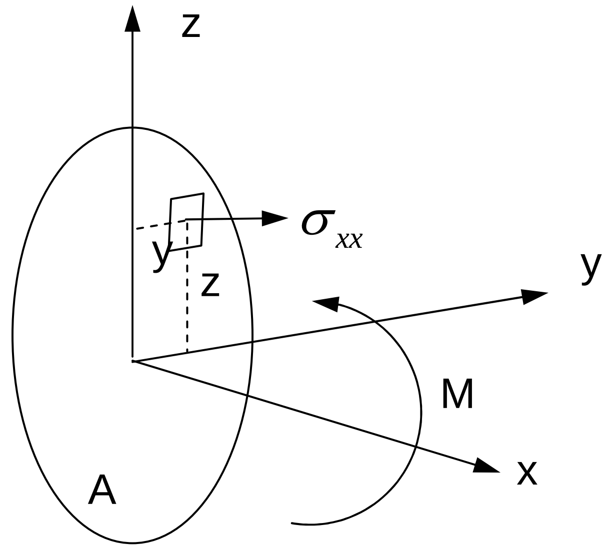
The stress-strain relations give

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\cancel{\sigma_{yy}} + \cancel{\sigma_{zz}}\right)\right]$$

$$\sigma_{xx} = -E\,z\frac{d^2 w}{dx^2}$$

$$\sigma_{xz} = G\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = G\left(\frac{dw}{dx} - \frac{dw}{dx}\right) = 0$$

$$\sigma_{xx} = -E\,z\,\frac{d^2 w}{dx^2}$$



$$M = -\int_A \sigma_{xx}\, z\, dA$$

$$= E\,\frac{d^2 w}{dx^2}\int_A z^2 dA$$

$$= EI\,\frac{d^2 w}{dx^2} \quad \Rightarrow \quad \sigma_{xx} = -\frac{M\,z}{I}$$

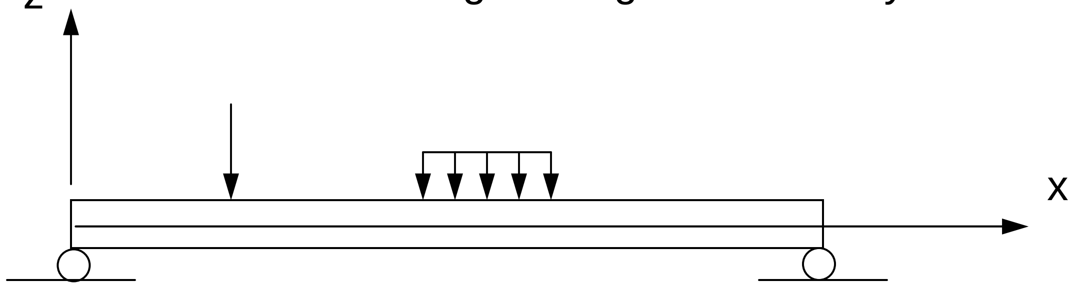
$$\int_A \sigma_{xx} dA = 0 \quad \Rightarrow \quad \int_A z dA = 0$$

neutral axis is at centroid

$$\int_A \sigma_{xx} y\, dA = 0 \quad \Rightarrow \quad \int_A y\, z\, dA = 0$$

cross-section must be symmetric

Engineering Beam Theory

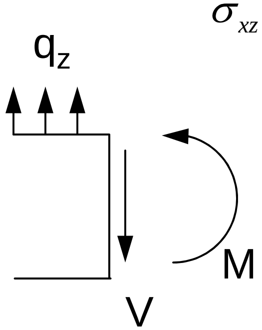


Let $\sigma_{xx} = -\dfrac{M(x)z}{I}$

$\sigma_{xz} = -\dfrac{V(x)Q(z)}{I\,t(z)}$

[Note: we still have $u_x = -z\dfrac{dw}{dx}$ $\left(\psi = \dfrac{dw}{dx}\right)$ so that

$\sigma_{xz} = 0$ (inconsistent)]



$$\frac{dM}{dx} = V(x)$$

$$\frac{dV}{dx} = q_z(x)$$

$$M = EI\frac{d^2w}{dx^2} \quad \Longrightarrow \quad EI\frac{d^4w}{dx^4} = q_z(x)$$

q_z … applied force/unit length on beam in z-direction

$$\frac{dM}{dx} = V\left(x\right)$$

$$\frac{dV}{dx} = q_z\left(x\right)$$

How are these internal force and bending moment equilibrium relations related to our local equilibrium equations?

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

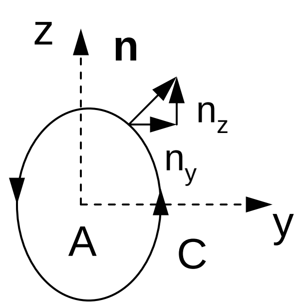
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

multiply by z and integrate over the cross-section, A

$$\int_A z \frac{\partial \sigma_{xx}}{\partial x} dA + \int_A z \left(\frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \right) dA = 0$$

or, equivalently

$$\underbrace{\frac{d}{dx} \int_A z \sigma_{xx} dA}_{-M(x)} + \int_A \left[\frac{\partial}{\partial y}\left(z\sigma_{xy} \right) + \frac{\partial}{\partial z}\left(z\sigma_{xz} \right) \right] dA - \underbrace{\int_A \sigma_{xz} dA}_{-V(x)} = 0$$

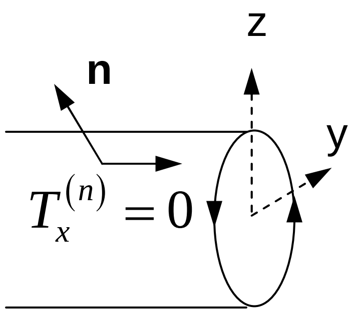


$$\int_A \frac{\partial f}{\partial y} dA = \oint_C f\, n_y\, ds$$

$$\int_A \frac{\partial f}{\partial z} dA = \oint_C f\, n_z\, ds$$

Gauss' theorem (2-D)

$$-\frac{dM}{dx}+\oint_C z\underbrace{\left(n_y\sigma_{xy}+n_z\sigma_{xz}\right)}_{T_x^{(n)}}ds+V\left(x\right)=0$$



$$T_x^{(n)}=0 \quad \Rightarrow \quad \frac{dM}{dx}=V\left(x\right)$$

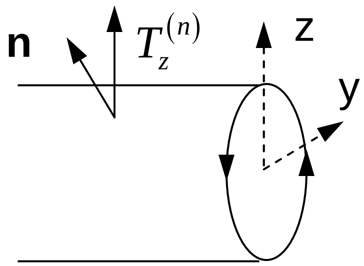
Now, consider

$$\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}=0$$

integrating over A

$$\frac{d}{dx}\underbrace{\int_A \sigma_{xz}dA}_{-V(x)}+\int_A\left(\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right)dA=0$$

$$-\frac{dV}{dx}+\oint_C \underbrace{\left(\sigma_{yz}n_y+\sigma_{zz}n_z\right)}_{T_z^{(n)}} ds = 0$$



$$\oint_C T_z^{(n)} ds = q_z\left(x\right)$$

applied force/unit length in z-direction

$$\Rightarrow \qquad \frac{dV}{dx} = q_z\left(x\right)$$

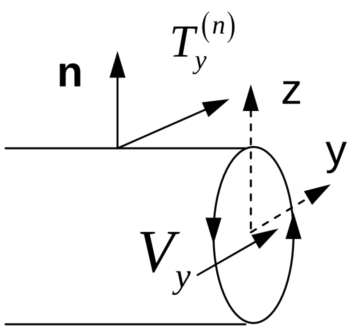
Last remaining equilibrium equation is:

$$\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{yz}}{\partial z}=0$$

Integrating over A gives

$$\frac{d}{dx}\underbrace{\int_A \sigma_{xy}\,dA}_{V_y}+\int_A\left(\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}\right)dA=0$$

$$\frac{dV_y}{dx}+\oint_C \underbrace{\left(\sigma_{yy}n_y+\sigma_{yz}n_z\right)}_{T_y^{(n)}}ds=0$$



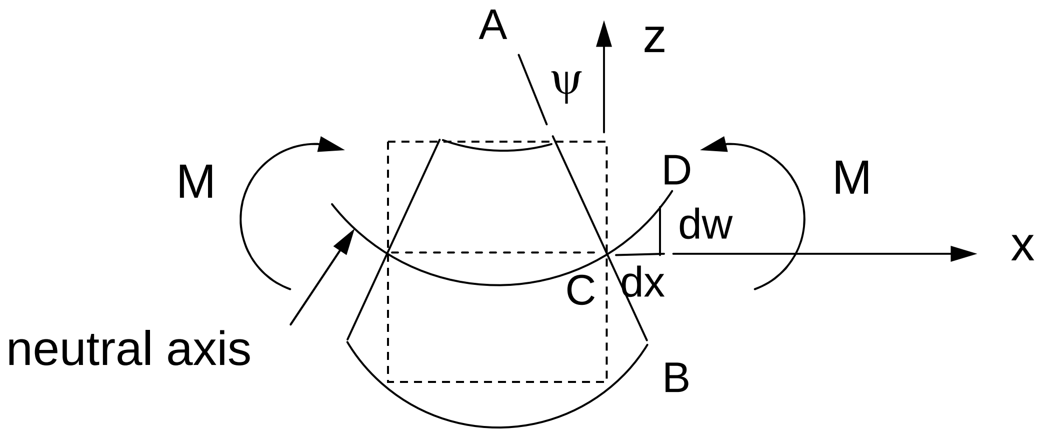
$$\oint_C T_y^{(n)}ds=q_y\left(x\right)$$ applied force/unit length in y-direction

$$\frac{dV_y}{dx}=-q_y\left(x\right)$$

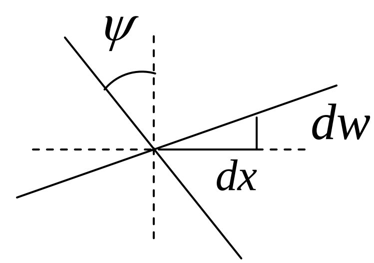
which is identically satisfied if

$$V_y=0,\;T_y^{(n)}=0$$

Timoshenko Beam Theory



$$u_x = -z\,\psi(x) \qquad \psi(x) \neq \frac{dw}{dx}$$



$$\sigma_{xx} = -E\,z\,\frac{d\psi}{dx}$$

$$\sigma_{xz} = G\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = G\left(-\psi(x) + \frac{dw}{dx}\right)$$

better than Euler/Bernoulli but still a constant across the cross-section so introduce a form factor κ^2

$$\sigma_{xx} = -E\,z\,\frac{d\psi}{dx}$$

$$\sigma_{xz} = \kappa^2 G\left(-\psi(x) + \frac{dw}{dx}\right)$$

$$\sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = 0$$

For bending moment and shear force

$$M = -\int_A z\sigma_{xx}\,dA = E\,\frac{d\psi}{dx}\int_A z^2\,dA = EI\,\frac{d\psi}{dx}$$

$$V = -\int_A \sigma_{xz}\,dA = -\kappa^2 G\left(-\psi + \frac{dw}{dx}\right)\int_A dA$$

$$= -\kappa^2 G\left(-\psi + \frac{dw}{dx}\right)A = -\sigma_{xz}A$$

Timoshenko Beam theory

$$M = EI\frac{d\psi}{dx}$$

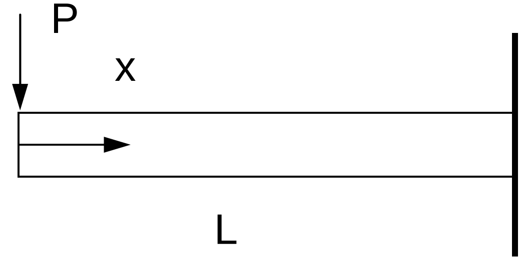
$$\frac{dw}{dx} - \psi\left(x\right) = -\frac{V\left(x\right)}{\kappa^2 GA}$$

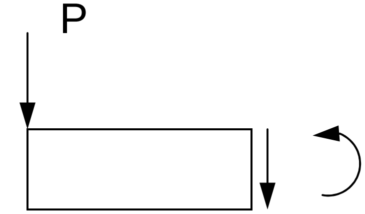
Euler-Bernoulli Theory

$$M = EI\frac{d^2 w}{dx^2}$$

$$\psi = \frac{dw}{dx}$$

Example:





$$EI\frac{d\psi}{dx} = -Px$$

$$EI\psi = -\frac{Px^2}{2} + C_1$$

$$\psi\left(L\right) = 0 \quad \Rightarrow \quad \psi = \frac{P\left[L^2 - x^2\right]}{2EI}$$

rotation of the beam cross-section

This gives $\sigma_{xx} = \frac{Pxz}{I}$ (same as ordinary beam theory)

$$\frac{dw}{dx} = -\frac{V}{\kappa^2 GA} + \psi$$ slope of neutral axis

$$= \frac{P}{\kappa^2 GA} + \frac{P\left(L^2 - x^2\right)}{2EI}$$

Integrating

$$w = \frac{Px}{\kappa^2 GA} + \frac{P\left(L^2 x - x^3/3\right)}{2EI} + C_2$$

$$w(L) = 0 \quad \Rightarrow \quad \frac{PL}{\kappa^2 GA} + \frac{PL^3}{3EI} + C_2 = 0$$

which gives $C_2 = w(0) = -\frac{PL^3}{3EI} - \frac{PL}{\kappa^2 GA}$

deflection due to: bending shear

$$w(0) = \frac{-PL^3}{3EI}\left[1 + \frac{3E}{\kappa^2 G}\frac{I}{AL^2}\right]$$

For a rectangular section of base b and height h $\frac{I}{AL^2} = \frac{1}{12}\left(\frac{h}{L}\right)^2$